Seat No. _______________

HA-003-1164002

M. Sc. (Sem. IV) Examination

April - 2023

Mathematics : CMT-4002

(Integration Theory)

Faculty Code : 003
Subject Code : 1164002

Time : $2\frac{1}{2}$ Hours / Total Marks : **70**

Instructions: (1) There are total five questions.
(2) All questions are mandatory.
(3) Each question carries equal marks.

1 Answer any seven of the following: **7×2=14**

(1) Define with example: Measure on measurable space.

(2) Define Positive and Negative set with respect to a signed measure.

(3) Show that, every measurable subset of a negative set with respect to signed measure on measurable space is a negative set.

(4) If $(A_i, B_i), i = 1, 2$ are Hahn-decomposition of X with respect to signed measure μ then show that, $A_1 \Delta A_2$ is null sets with respect to μ.

(5) Define with example: A measure absolutely continuous with respect to another measure.

(6) Let γ_1, γ_2 and μ be measures on measurable space $(X, \mathcal{A})$ and $\gamma_1 \perp \mu$, $\gamma_2 \perp \mu$ then show that, $c_1 \gamma_1 + c_2 \gamma_2 \perp \mu$.

(7) Define with example: Complete measure.

(8) State, Riesz-Representation Theorem for bounded linear functional on $L^p(\mu)$.

(9) State, Fubini's Theorem without proof.

(10) Define Baire sets in a locally compact Hausdorff space.

2 Answer any two of the following: **2×7=14**

(1) Let μ_1, μ_2 be two measures on measurable space $(X, \mathcal{A})$ such that at least one of them is finite then show that, $\mu : \mathcal{A} \to [-\infty, \infty]$ defined by $\mu = \mu_1 - \mu_2$ is a signed measure on $(X, \mathcal{A})$.

(2) State and prove, Hahn Decomposition Theorem.

(3) Let $(X, \mathcal{A})$ be a measurable space, $D \subseteq \mathbb{R}$ be dense in $\mathbb{R}$ and $B_\alpha, \alpha \in D$ be measurable such that $B_\alpha \subset B_\beta, \forall \alpha, \beta \in D$ with $\alpha < \beta$ then show that, there exists a unique measurable function $f : X \to [-\infty, \infty]$ such that $f(x) \le \alpha, \forall x \in \beta_\alpha$ and $f(x) \ge \alpha, \forall x \in X \setminus \beta_\alpha, \forall \alpha \in D$.

3 Answer the following: **2×7=14**

(1) State and prove, Lebesgue decomposition theorem.

(2) State and prove, Radon-Nikodym theorem for signed measures.

OR

3 Answer the following : **2×7=14**

(1) State and prove, Caratheodory Extension Theorem.

(2) Let $(X, \mathcal{A}, \mu)$ be a complete measure space and $1 \le p < \infty$. Show that, S is dense in $L^p(\mu)$. Where

$S = \{s \,/\, s \text{ is simple measurable on } X \;\&\; \mu(\{x \in X \,/\, s(x) \neq 0\}) < \infty\}$

4 Answer the following: **2×7=14**

(1) Let $F:\mathbb{R}\to\mathbb{R}$ be monotonically increasing and continuous on the right and $(a,b]\subset \bigcup_{n=1}^{\infty}(a_n,b_n]$ then show that,

$$F(b)-F(a)\le \sum_{n=1}^{\infty}(F(b_n)-F(a_n))$$

Where $F(-\infty)=\lim_{x\to-\infty}F(x)$ and $F(\infty)=\lim_{x\to\infty}F(x)$.

(2) Let μ be a measure on an algebra A of subset of a set X and μ^* be the outer measure on X induced by μ then prove that, every element $E\in\mathcal{A}$ is μ^* measurable.

5 Answer any two of the following. **2×7=14**

(1) Let $(X\times Y,\ F,\ \mu\times\gamma)$ be the product measure space of two σ-finite complete measure spaces. $\Re$ be the semi algebra of measurable rectangles in $X\times Y$, $E\in\Re_{\sigma\delta}$ and $x\in X$ then show that, E_x is a measurable subset of Y.

(2) Describe by example that the hypothesis "the non negativity of f" in Tonnelli's theorem can not be dropped.

(3) Let X be a locally compact Hausdorff space. Then show that, $C_c(x)=\{f:X\to\mathbb{R}\,/\,f$ is continuous and *Supp* f is compact in $X\}$ is vector space over $\mathbb{R}$ with respect to point wise addition and scalar multiplication.

(4) Describe by example that Hahn decomposition of X is not unique with respect to a signed measure on $(X,\ \mathcal{A})$.
